A SIMPLIFIED SOLUTION OF THE PROBLEM OF ELASTOPLASTIC EQUILIBRIUM OF A CYLINDER IN A NONUNIFORM TEMPERATURE FIELD

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A solution is given of the problem of thermal stresses in a solid circular cylinder, allowing for compressibility of the material and the Mises plasticity condition. It is shown that the radial and annular stresses are determined by simplified formulas with sufficient accuracy for engineering purposes.

In the analysis of cylindrical bodies in the stressed condition in a nonuniform temperature field, a plasticity condition of the simplest type is used in some papers [1, 2].

$$\sigma_\theta - \sigma_r = -2k. \tag{1}$$

Here the radial and annular stresses in the plastic strain region are statically determinate and are found by integration of the equation of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{2}$$

In particular, when there is plastic strain in the peripheral layers of the section of a cylinder heated from within and there are no external loads, i.e., when

$$\sigma_r|_{r=R} = 0, \tag{3}$$

the radial and annular stresses are determined by the relations

$$\sigma_r = 2\int_r^R \frac{kdr}{r}, \tag{4}$$

$$\sigma_\theta = 2\left(\int_r^R \frac{kdr}{r} - k\right). \tag{5}$$

In the elastic strain region ($0 \le r \le r_0$), assuming no change in the elastic characteristics of the material, the stress components are determined by the relation [1]

$$\sigma_r^l = \frac{\alpha E}{1-\mu}\left(\frac{1}{r_0^2}\int_0^{r_0} Trdr - \frac{1}{r^2}\int_0^r Trdr\right) + \sigma_{r0}^S, \tag{6}$$

$$\sigma_\theta^l = \frac{\alpha E}{1-\mu}\left(\frac{1}{r_0^2}\int_0^{r_0} Trdr + \frac{1}{r^2}\int_0^r Trdr - T\right) + \sigma_{r0}^S, \tag{7}$$

the continuity of radial and annular stresses being maintained at the boundary between the regions, i.e.,

$$\sigma_{r0}^S = \sigma_{r0}^l, \quad \sigma_{\theta 0}^S = \sigma_{\theta 0}^l \quad \text{for } r = r_0. \tag{8}$$

The determination of longitudinal stresses in a cylinder requires more detailed study of the interrelation of strains and stresses in elastoplastic deformation.

In using the theory of small elastoplastic strains, the strain and stress components are interrelated by the Hencky equations:

$$\begin{aligned} \varepsilon_r &= \psi(\sigma_r - \sigma)/2G + N\sigma + \alpha T, \\ \varepsilon_\theta &= \psi(\sigma_\theta - \sigma)/2G + N\sigma + \alpha T, \\ \varepsilon_z &= \psi(\sigma_z - \sigma)/2G + N\sigma + \alpha T. \end{aligned} \tag{9}$$

From the last equation of (9) we find

$$\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta) + \frac{3G}{\psi}(\varepsilon_z - N\sigma - \alpha T). \tag{10}$$

For a compressible material, the plasticity modulus ψ is determined by the type of stress condition, which makes it more difficult to find σ_z from (10). In this connection, in order to find σ_z, additional simplifications in the formulation of the problem are made.

Use is made in [3, 4] of the assumption of incompressibility of the material in the plastic strain region, allowing explicit expressions to be obtained for the plasticity modulus and the longitudinal strain.

It has been shown in [4] that when the Tresk condition is used and the material is assumed to be incompressible ($\mu = 0.5$), the plasticity modulus is determined by the equation

$$\psi(r) = \frac{r_0^2}{r^2} + \frac{E}{k}\left[T(r) - \frac{r_0^2}{r^2}T(r_0)\right] - \frac{2}{r_0^2}\int_{r_0}^r Trdr. \tag{11}$$

The longitudinal strain is

$$\varepsilon_z = \left[0.5\alpha ET(r_0) - \sigma(r_0) + k\left(\frac{R^2}{r_0^2} - 1\right) - \frac{2}{r_0^2}\int_{r_0}^R \sigma_r rdr + \right.$$
$$\left. + \frac{2\alpha E}{r_0^2}\int_{r_0}^R \frac{Trdr}{\psi}\right]\left[E\left(1 + \frac{2}{r_0^2}\int_{r_0}^R \frac{rdr}{\psi}\right)\right]^{-1}. \tag{12}$$

If, in addition to the assumption of incompressibility of the material, a simplified expression for the temperature field is used in the form

$$T(r) = \Delta T_m r^2/R^2, \tag{13}$$

we find from (11) and (12) that

$$\psi = \left(\frac{r}{r_0}\right)^2, \quad \varepsilon_z = \frac{\alpha\Delta T_m}{2}\,\frac{2 - r_0^2/R^2}{1 + 2\ln(R/r_0)}. \tag{14}$$

When the distribution is parabolic, expressions (6) and (7) take the form

$$\begin{aligned} \sigma_r^l &= \frac{\alpha E\Delta T_m}{4(1-\mu)}\left(1 - \frac{r^2}{r_0^2}\right) + \sigma_{r0}^S, \\ \sigma_\theta^l &= \frac{\alpha E\Delta T_m}{4(1-\mu)}\left(1 - 3\frac{r^2}{r_0^2}\right) + \sigma_{r0}^S. \end{aligned} \tag{15}$$

Thus, by making a number of assumptions, we may obtain very simple expressions for the basic parameters to be calculated. It should be noted that the plasticity condition in the form of Eq. (1) may be used only when the longitudinal stresses (σ_z) are intermediate in magnitude between the radial (σ_r) and the annular (σ_θ) stresses. In cases when this condition is not evident, it is necessary to use the more general plasticity condition of Mises [5]

$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2=6k^2. \quad (15')$$

The solution is given below for the state of stress of a solid circular cylinder in a nonuniform temperature field. Compressibility of the cylinder material is allowed for ($N \neq 0$ in the Hencky equations) when using the Mises plasticity condition (15') and a constant yield point (k) is assumed. The solution is derived by using the trigonometrical formulation put forward in (6) during examination of the state of stress of a hollow cylindrical tube subjected to internal and external pressures.

We express the stress components by the equations

$$\sigma_r^S=k(c-\alpha T)/M+k(\omega-\sin\varphi), \quad (16)$$

$$\sigma_\theta^S=k(c-\alpha T)/M+k(\omega+\sin\varphi), \quad (17)$$

$$\sigma_z^S \quad k(c-\alpha T)/M+k(\omega-\sqrt{3}\cos\varphi), \quad (18)$$

where $M = 2(1+\mu)/(1-2\mu) = \text{const}$ and $c = \text{const}$.

It is not hard to verify that Eqs. (16)–(18) satisfy the Mises plasticity condition (15'). Allowing for the condition $\varepsilon_z = \text{const}$, the equation of strain compatibility may be written in the form

$$\frac{\partial(\varepsilon_z-\varepsilon_\theta)}{\partial r}+\frac{\varepsilon_r-\varepsilon_\theta}{r}=0,$$

and, since

$$\varepsilon_z-\varepsilon_\theta=\frac{\psi}{2G}(\sigma_z^S-\sigma_\theta^S), \quad \varepsilon_r-\varepsilon_\theta=\frac{\psi}{2G}(\sigma_r^S-\sigma_\theta^S),$$

we obtain

$$\psi\frac{\partial(\sigma_z^S-\sigma_\theta^S)}{\partial r}+(\sigma_z^S-\sigma_\theta^S)\frac{\partial\psi}{\partial r}-\psi\frac{\sigma_\theta^S-\sigma_r^S}{r}=0. \quad (19)$$

The last of the Hencky equations leads to the relation

$$\psi(\sigma_r^S+\sigma_\theta^S-2\sigma_z^S)=2M\left[3\sigma^S-\frac{3k}{M}(\varepsilon_z-\alpha T)\right]. \quad (20)$$

Substituting expressions (9)–(11) into (2), (12), and (13), we obtain the relation

$$\omega=\frac{\cos\varphi}{\sqrt{3}}\left(\frac{\psi}{M}+1\right), \quad (21)$$

and the differential equations

$$\frac{\partial\varphi}{\partial r}=\frac{2\sin\varphi}{r(\cos\varphi-\sqrt{3}\sin\varphi)}-\frac{1}{\psi}\frac{\partial\psi}{\partial r}\frac{\sqrt{3}\cos\varphi+\sin\varphi}{\cos\varphi-\sqrt{3}\sin\varphi}, \quad (22)$$

$$\frac{\partial\varphi}{\partial r}=-\frac{2\sin\varphi}{r[(1+\psi/M)\sin\varphi/\sqrt{3}+\cos\varphi]}-\frac{\alpha\,\partial T/\partial r}{M[(1+\psi/M)\sin\varphi/\sqrt{3}+\cos\varphi]}. \quad (23)$$

Equating the right sides of (22) and (23), we have

$$\frac{\partial\psi}{\partial r}=\left[(\sqrt{3}\cos\varphi-3\sin\varphi)\alpha\frac{\partial T}{\partial r}-2M\sin^2\varphi(4+\psi/M)/r\right]\times$$
$$\times\{\cos\varphi(\cos\varphi-\sqrt{3}\sin\varphi)+(\sqrt{3}\cos\varphi+\sin\varphi)\times$$
$$\times[\sqrt{3}\cos\varphi+\sin\varphi(1+\psi/M)]M/\psi\}^{-1}. \quad (24)$$

The system of equations (22) and (24) may be solved by the method of finite differences. Replacing the derivatives by finite increment relations, we obtain a system of finite difference equations:

$$\varphi_{k+1}=\varphi_k-\frac{2\Delta r}{r_k}\frac{\sin\varphi_k}{\cos\varphi_k-\sqrt{3}\sin\varphi_k}-\left(\frac{\psi_{k+1}}{\psi_k}-1\right)\frac{\sqrt{3}\cos\varphi_k+\sin\varphi_k}{\cos\varphi_k-\sqrt{3}\sin\varphi_k},$$

$$\psi_{k+1}=$$
$$=\psi_k+\Delta r[\alpha|\partial T/\partial r|_k(\sqrt{3}\cos\varphi_k-3\sin\varphi_k)-2M\sin^2\varphi_k\times$$
$$\times(4+\psi_k/M)/r_k]\{\cos\varphi_k(\cos\varphi_k-\sqrt{3}\sin\varphi_k)+$$
$$+(M/\psi_k)\times(\sqrt{3}\cos\varphi_k+\sin\varphi_k)[\sqrt{3}\cos\varphi_k+$$
$$+\sin\varphi_k(1+\psi_k/M)]\}^{-1}. \quad (25)$$

The system (25) allows us to determine values of the functions $\varphi(r)$ and $\psi(r)$ in succession, step by step (Δr), beginning from the boundary r_0 of regions of elastic and plastic strain. The boundary values of the functions when $r = r_0$ are found in the following way.

It follows from (16) and (17) that

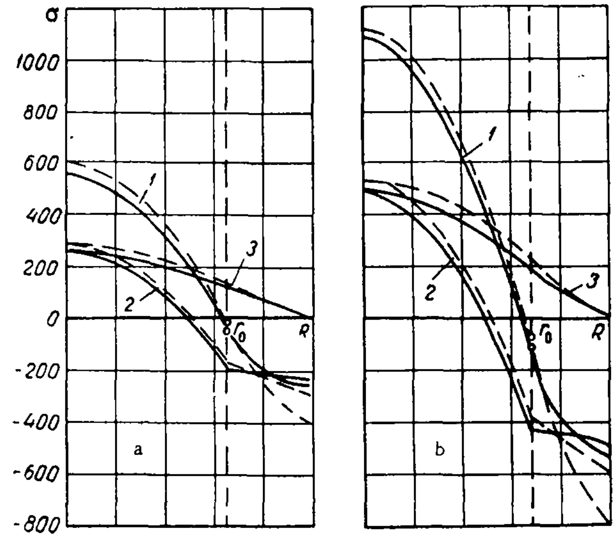
$$\sigma_\theta^S-\sigma_r^S=2k\sin\varphi,$$
$$\varphi_0=\arcsin\left|\frac{\sigma_\theta^S-\sigma_r^S}{2k}\right|_{r=r_0}. \quad (26)$$

The difference of the stresses on the right side of (26) may be found from the continuity condition (8) at the boundary of the region, taking expressions (15) into account:

$$\sigma_\theta-\sigma_r=\frac{\alpha E}{1-\mu}\left[T(r_0)-\frac{2}{r_0^2}\int_0^{r_0}Trdr\right]=-\frac{\alpha E\Delta T_m}{2(1-\mu)}=-2k,$$
$$\varphi_0=-\pi/2.$$

The modulus of plasticity at the boundary of the region is equal to unity i.e., $\psi_0 = 1$. The solution of the system (25) has been carried out for values $\varphi_0 = -\pi/2$ and $\psi_0 = 1$, the region of plastic strain ($r_0 \le r \le R$) being divided into 20 intervals of equal width $\Delta r^S = (R - r_0)/20$.

After finding the functions φ and ψ, the stresses in the plastic region were calculated* from (16)–(18), and those in the region of elastic strain from (15), the constant c being determined from (16) allowing for the boundary condition (3).



Thermal stress distribution along the cylinder radius (MN/m^2): a) $\alpha = 10^{-5}$ deg^{-1}; E = $19.6 \cdot 10^4$ MN/m^2; μ = = 0.25; k = 147 MN/m^2 (steel); ΔT_m = 540 deg; b) α = = $1.5 \cdot 10^{-5}$ deg^{-1}; E = $22 \cdot 10^4$ MN/m^2; μ = 0.25; k = = 294 MN/m^2 (steel); ΔT_m = 600 deg; 1—σ_z; 2—σ_θ; 3—σ_r.

The results are shown in the figure by solid lines for two sets of initial parameters and compared with those of calculation (dotted lines) according to the simplified solution, based on the assumptions of incompressibility of the material in the plastic strain region and the Tresk plasticity condition.

It may be seen from the figure that the stress results from the two methods do not differ greatly. In particular, on the cylinder axis the stress components found by allowing for compressibility of the material are not more than 3–4% less than the corresponding results of the simplified solution, when μ^S = 0.5 and N = 0. The discontinuity in longitudinal stress distribution along the cylinder radius at $r = r_0$ (using the simplified solution), caused by the difference in Poisson's ratio in the two regions, does not exceed 5% of the maximum values of σ_z.

The analysis made allows us to conclude that when there is a comparatively small region of plastic strain encompassing the peripheral layers of the cylinder section, the determination of thermal stresses in the axial region of the cylinder (i.e., in the elastic strain region) on the assumption of incompressibility of the material and the Tresk flow condition is quite justified.

It should additionally be noted that the above method of solving the problem by using a trigonometrical formulation is not sufficiently universal.

From comparison of the system (16)–(18) with (8), we derive the conditions $c = \varepsilon_z$. On the other hand, the magnitude of the longitudinal strain ε_z was found from the condition that the moment of the longitudinal forces is zero, i.e.,

$$\int_0^R \sigma_z r dr = 0. \tag{27}$$

In view of this, among the test computational conditions chosen (and presented in the figure), we included conditions for which the choice of the constant $c = \varepsilon_z$ was associated with fulfillment of boundary conditions (3) and (27). Thus the method of solution described, which takes account of the compressibility of the material, allows us to determine uniquely only the radial and annular stresses, while additional conditions connected with the determination of the longitudinal strain ε_z are required for finding the longitudinal stresses.

In regard to this difficulty, Hill's remarks [5] should be noted. He assumed that the longitudinal stresses in a cylindrical tube are determined by the strain history and should therefore be found by means of a system of Reuss kinetic equations instead of a system of Hencky equations.

NOTATION

σ_r, σ_θ, σ_z) radial, annular, and longitudinal components of the stress tensor; ε_r, ε_θ, ε_z) the same, for the strain tensor; T) excess temperature; α) coefficient of linear expansion; G) shear modulus; E) Young's modulus; k) yield point in shear; σ^S) the same, in tension; μ) Poisson's ratio; r) coordinate along the cylinder radius; r_0) radius of the boundary between regions of elastic and plastic strain; R) cylinder radius; ψ) modulus of plasticity.

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*Using the electronic computer "Minsk-1."